SUBSPACES OF ALMOST DAUGAVET SPACES

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ABSTRACT. We study the almost Daugavet property, a generalization of the Daugavet property. It is analysed what kind of subspaces and sums of Banach spaces with the almost Daugavet property have this property as well. The main result of the paper is: if Z is a closed subspace of a separable almost Daugavet space X such that the quotient space X/Z contains no copy of ℓ_1, then Z has the almost Daugavet property, too.

1. INTRODUCTION

V. Kadets, V. Shepelska, and D. Werner characterized in [2] a class of separable Banach spaces in three very different ways.

Theorem 1.1. *For a separable Banach space X the following conditions are equivalent:*

(i) *S_X does not admit a finite ε-net consisting of elements of S_X for any $\varepsilon < 2$.*

(ii) *there is a sequence $(e_n)_{n \in \mathbb{N}}$ in B_X such that for every $x \in X$*

$$\lim_{n \to \infty} \|x + e_n\| = \|x\| + 1.$$

(iii) *there is a norming subspace $Y \subset X^*$ such that the equation*

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{1.1}$$

holds true for every rank-one operator $T : X \to X$ of the form $T = y^* \otimes x$, where $x \in X$ and $y^* \in Y$.

Recall that a subspace $Y \subset X^*$ is said to be norming if for every $x \in X$

$$\sup_{y^* \in S_Y} |y^*(x)| = \|x\|.$$

Condition (i) can be rephrased using the so-called *thickness* of a Banach space X, that was introduced by R. Whitley in [5]. We call a set F an inner ε-net of S_X if $F \subset S_X$ and for every $x \in S_X$, there is an $y \in F$ with $\|x - y\| \leq \varepsilon$. Then the thickness $T(X)$ of a Banach space X is defined by

$$T(X) = \inf \{\varepsilon > 0 \colon \text{there exists a finite inner } \varepsilon\text{-net of } S_X\}.$$

Essentially, the thickness is a kind of inner measure of non-compactness of the unit sphere S_X.

R. Whitley showed that $1 \leq T(X) \leq 2$ if X is infinite-dimensional, in particular that $T(l_p) = 2^{1/p}$ for $1 \leq p < \infty$ and $T(C(K)) = 2$ if K has no isolated points.

So condition (i) means that $T(X) = 2$.

Condition (ii) links the investigation to the theory of types, that were used by J.-L. Krivine and B. Maurey in [4]. A type on a separable Banach space is a function of the form

$$\tau(x) = \lim_{n \to \infty} \|x + e_n\|$$

for some bounded sequence $(e_n)_{n\in\mathbb{N}}$. In [4] a convolution of types is defined and by means of this convolution the notion of ℓ_1-types is introduced. A special ℓ_1-type is generated by a sequence $(e_n)_{n\in\mathbb{N}}$ satisfying

$$\tau(x) = \lim_{n\to\infty} \|x + e_n\| = \|x\| + 1.$$

We call a sequence $(e_n)_{n\in\mathbb{N}}$ in B_X satisfying this equation a *canonical ℓ_1-type sequence*. The following proposition shows us that a canonical ℓ_1-type sequence (or a suitable subsequence) almost behaves like the canonical base of ℓ_1.

Proposition 1.2. *Let X be a separable Banach space, $E_1 \subset E_2 \subset E_3 \subset \dots$ a sequence of finite-dimensional subspaces whose union is dense in X, and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers that converges monotonically towards zero. If $(e_n)_{n\in\mathbb{N}}$ is a canonical ℓ_1-type sequence then we can pass to a subsequence $(f_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$ such that for every $x \in E_n$ and $y = \sum_{k=n+1}^{M} \lambda_k f_k$*

$$\|x + y\| \geq (1 - \varepsilon_n) \|x\| + \sum_{k=n+1}^{M} (1 - \varepsilon_{k-1}) |\lambda_k|.$$

Proof. Fix a sequence of positive real numbers $(\delta_n)_{n\in\mathbb{N}}$ with $\prod_{k=n}^{\infty} (1 - \delta_k) \geq 1 - \varepsilon_n$ for all $n \in \mathbb{N}$. There is a subsequence $(f_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$ such that for every $x \in \operatorname{span}(E_n \cup \{f_1, \dots, f_n\})$ and $\lambda \in \mathbb{K}$

$$\|x + \lambda f_{n+1}\| \geq (1 - \delta_n) (\|x\| + |\lambda|).$$

Then we have for every $x \in E_n$ and $y = \sum_{k=n+1}^{M} \lambda_k f_k$ as a consequence of the choice of $(\delta_n)_{n\in\mathbb{N}}$

$$\|x + y\| = \left\| x + \sum_{k=n+1}^{M} \lambda_k f_k \right\| \geq (1 - \delta_{M-1}) \left\| x + \sum_{k=n+1}^{M-1} \lambda_k f_k \right\| + (1 - \delta_{M-1}) |\lambda_M|$$

$$\geq \dots \geq \prod_{k=n}^{M-1} (1 - \delta_k) \|x\| + \sum_{k=n+1}^{M} (1 - \delta_{k-1}) |\lambda_k|$$

$$\geq (1 - \varepsilon_n) \|x\| + \sum_{k=n+1}^{M} (1 - \varepsilon_{k-1}) |\lambda_k|. \qquad \square$$

Condition (iii) links the investigation to the theory of Banach spaces with the *Daugavet property* introduced by V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner in [3]. We say that a Banach space X has the *Daugavet property with respect to Y* if the *Daugavet equation* (1.1) holds true for every rank-one operator $T : X \to X$ of the form $T = y^* \otimes x$, where $x \in X$ and $y^* \in Y$. A Banach space X is called an *almost Daugavet space* or a space with the *almost Daugavet property* if it has the Daugavet property with respect to some norming subspace $Y \subset X^*$. This definition, that was introduced in [1], is a generalization of the Daugavet property. Recall that a Banach space X has the Daugavet property if it has the Daugavet property with respect to X^*.

In the following section we study some structural properties of the class of almost Daugavet spaces, and our main tool will be theorem 1.1. We show that the ℓ_1- and ℓ_∞-sum of separable almost Daugavet spaces are almost Daugavet spaces. Concerning subspaces, we show results about L-summands and M-ideals. But the main result will be that the almost Daugavet property of a separable Banach space X is inherited by a closed subspace Z if the quotient space X/Z contains no copy of ℓ_1.

Special thanks go to V. Kadets for giving an important hint to achieve this result in such a general setting.

2. SUBSPACES AND SUMS OF ALMOST DAUGAVET SPACES

2.1. **Sums.** The almost Daugavet property depends very much on the norm of a space. So it can't be expected that arbitrary sums of almost Daugavet spaces still have the almost Daugavet property. But there are the following results concerning ℓ_1- and ℓ_∞-sums.

Proposition 2.1. *If X and Y are two separable almost Daugavet spaces, then $X \oplus_\infty Y$ is an almost Daugavet space as well.*

Proof. As a consequence of theorem 1.1, X as well as Y contains a canonical ℓ_1-type sequence $(e_n)_{n\in\mathbb{N}}$ resp. $(f_n)_{n\in\mathbb{N}}$. Every member of $((e_n, f_n))_{n\in\mathbb{N}}$ is an element of $B_{X\oplus_\infty Y}$ and for $(x, y) \in X \oplus_\infty Y$ we have

$$\lim_{n\to\infty} \|(x, y) + (e_n, f_n)\| = \lim_{n\to\infty} \max\{\|x + e_n\|, \|y + f_n\|\}$$
$$= \max\left\{\lim_{n\to\infty} \|x + e_n\|, \lim_{n\to\infty} \|y + f_n\|\right\}$$
$$= \max\{\|x\| + 1, \|y\| + 1\}$$
$$= \max\{\|x\|, \|y\|\} + 1 = \|(x, y)\| + 1.$$

So $((e_n, f_n))_{n\in\mathbb{N}}$ is a canonical ℓ_1-type sequence in $X \oplus_\infty Y$ and $X \oplus_\infty Y$ is an almost Daugavet space by theorem 1.1. $\qquad\square$

Proposition 2.2. *Let X and Y be separable Banach spaces. If X has the almost Daugavet property, then so does $X \oplus_1 Y$.*

Proof. Because of theorem 1.1, X contains a canonical ℓ_1-type sequence $(e_n)_{n\in\mathbb{N}}$. For $(x, y) \in X \oplus_1 Y$ we get

$$\lim_{n\to\infty} \|(x, y) + (e_n, 0)\| = \lim_{n\to\infty} (\|x + e_n\| + \|y\|) = \lim_{n\to\infty} \|x + e_n\| + \|y\|$$
$$= \|x\| + \|y\| + 1 = \|(x, y)\| + 1,$$

and thus $((e_n, 0))_{n\in\mathbb{N}}$ is a canonical ℓ_1-type sequence. Therefore $X \oplus_1 Y$ is an almost Daugavet space as a result of theorem 1.1. $\qquad\square$

2.2. **Subspaces.** Now we are interested in the question which subspaces of almost Daugavet spaces inherit the almost Daugavet property. Since the almost Daugavet property depends very much of the norm of a space, we are going to study subspaces fulfilling adequat conditions.

Proposition 2.3. *Let X be a separable almost Daugavet space, J an L-summand and $\widehat{J}$ the complementary L-summand (i.e., $X = J \oplus_1 \widehat{J}$). Then J or $\widehat{J}$ has the almost Daugavet property.*

Proof. As a result of theorem 1.1, it suffices to show that $T(J) = 2$ or $T(\widehat{J}) = 2$. Let us suppose that $T(J) < 2$ and $T(\widehat{J}) < 2$. Then there exist for an $\varepsilon > 0$ inner $(2 - 2\varepsilon)$-nets of S_J and $S_{\widehat{J}}$. Let $\{j_1, \ldots, j_l\}$ be such an inner $(2 - 2\varepsilon)$-net of S_J and $\{\widehat{j}_1, \ldots, \widehat{j}_m\}$ one of $S_{\widehat{J}}$. Furthermore, let $\{\lambda_1, \ldots, \lambda_n\}$ be an inner $(\varepsilon/2)$-net of $[0, 1]$.
As J and $\widehat{J}$ are L-summands, the unit sphere of X can be expressed by

$$S_X = \left\{\left(\lambda j, (1 - \lambda)\widehat{j}\right) : \lambda \in [0, 1], j \in S_J, \widehat{j} \in S_{\widehat{J}}\right\}.$$

Given $x = (\lambda j, (1 - \lambda)\widehat{j}) \in S_X$, we can pick $\lambda_0 \in \{\lambda_1, \ldots, \lambda_n\}$ with $|\lambda_0 - \lambda| \le \varepsilon/2$, $j_0 \in \{j_1, \ldots, j_l\}$ with $\|j_0 - j\| \le 2 - 2\varepsilon$ and $\widehat{j}_0 \in \{\widehat{j}_1, \ldots, \widehat{j}_m\}$ with $\|\widehat{j}_0 - \widehat{j}\| \le 2 - 2\varepsilon$.

Then we get with $y = (\lambda_0 j_0, (1-\lambda_0)\widehat{j_0})$

$$\|x - y\| = \|\lambda j - \lambda_0 j_0\| + \left\|(1-\lambda)\widehat{j} - (1-\lambda_0)\widehat{j_0}\right\|$$

$$= \|(\lambda - \lambda_0)j + \lambda_0(j - j_0)\| + \left\|(\lambda_0 - \lambda)\widehat{j} + (1-\lambda_0)(\widehat{j} - \widehat{j_0})\right\|$$

$$\leq |\lambda - \lambda_0|\left(\|j\| + \|\widehat{j}\|\right) + \left(\lambda_0\|j - j_0\| + (1-\lambda_0)\|\widehat{j} - \widehat{j_0}\|\right)$$

$$\leq \varepsilon + (2 - 2\varepsilon) = 2 - \varepsilon.$$

Thus $\left\{(\lambda j, (1-\lambda)\widehat{j}) \colon \lambda \in \{\lambda_1, \ldots, \lambda_n\}, j \in \{j_1, \ldots, j_l\}, \widehat{j} \in \{\widehat{j_1}, \ldots, \widehat{j_m}\}\right\}$ is an inner $(2-\varepsilon)$-net of S_X. But this is because of theorem 1.1 a contradiction to the assumption that X has the almost Daugavet property. $\qquad\square$

Recall that an M-ideal in a Banach space X is a closed subspace J such that X^* decomposes as $X^* = Y \oplus_1 J^\perp$ for some closed subspace Y of X^*, where $J^\perp = \{x^* \in X^* \colon x^*(x) = 0 \text{ for all } x \in J\}$.
In [3, Proposition 2.10.] it is shown that M-ideals of Banach spaces with the Daugavet property have the Daugavet property as well. The same proof works for almost Daugavet spaces.

Proposition 2.4. *Let X be a Banach space, Y a norming subspace of X^* and suppose that X has the Daugavet property with respect to Y. If $J \neq \{0\}$ is an M-ideal in X, then J has the Daugavet property with respect to $\{y^*|_J \colon y^* \in Y\}$. So J is an almost Daugavet space as well.*

On the other hand, almost Daugavet spaces are in a certain sense "big". They have thickness 2 and contain a copy of ℓ_1. So in the next theorem we consider "big" subspaces of an almost Daugavet space.

Theorem 2.5. *Let X be a separable almost Daugavet space and Z a closed subspace of X. If the quotient space X/Z contains no copy of ℓ_1, then Z has the almost Daugavet property, too.*

Proof. By theorem 1.1, it suffices to construct a canonical ℓ_1-type sequence in Z. Let $E_1 \subset E_2 \subset E_3 \subset \ldots$ be a sequence of finite-dimensional subspaces whose union is dense in X. Also, fix a sequence of positive real numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ that converges monotonically towards zero. As a consequence of theorem 1.1 and proposition 1.2, there is a canonical ℓ_1-type sequence $(e_n)_{n \in \mathbb{N}}$ in B_X satisfying the following condition: for every $x \in E_n$ and every $y = \sum_{k=n+1}^{M} \lambda_k e_k$ we have

$$\|x + y\| \geq (1 - \varepsilon_n)\|x\| + \sum_{k=n+1}^{M} (1 - \varepsilon_{k-1})|\lambda_k|. \tag{2.1}$$

So the space $\overline{\operatorname{span}}\{e_n \colon n \geq k\}$ is isomorphic to ℓ_1 for each $k \in \mathbb{N}$. Since X/Z contains no copy of ℓ_1, the quotient map $\pi : X \to X/Z$ fails to be bounded below on each subspace $\overline{\operatorname{span}}\{e_n \colon n \geq k\}$, i.e., for each $k \in \mathbb{N}$, there is no $C > 0$ with $C\|x\| \leq \|\pi(x)\|$ for all $x \in \overline{\operatorname{span}}\{e_n \colon n \geq k\}$.
As a result of this, there is a linear combination $\sum_{k=1}^{m_1} \lambda_k e_k$ with $\|\sum_{k=1}^{m_1} \lambda_k e_k\| = 1$ and $\|\pi(\sum_{k=1}^{m_1} \lambda_k e_k)\| \leq \varepsilon_1/4$. By the definition of the quotient norm, there is an $\hat{f_1} \in Z$ with $\|\hat{f_1} - \sum_{k=1}^{m_1} \lambda_k e_k\| \leq \varepsilon_1/2$, and we get $\|f_1 - \sum_{k=1}^{m_1} \lambda_k e_k\| \leq \varepsilon_1$ with $f_1 = \hat{f_1}/\|\hat{f_1}\|$. Going on like this, we get for each $n \in \mathbb{N}$ a linear combination $\sum_{k=m_{n-1}+1}^{m_n} \lambda_k e_k$ with $\|\sum_{k=m_{n-1}+1}^{m_n} \lambda_k e_k\| = 1$ and an element $f_n \in S_Z$ with $\|f_n - \sum_{k=m_{n-1}+1}^{m_n} \lambda_k e_k\| \leq \varepsilon_n$.
It remains to show that $(f_n)_{n \in \mathbb{N}}$ is a canonical ℓ_1-type sequence. Given $x \in X$ and $\varepsilon > 0$, we can assume without loss of generality that $x \in E_{\hat{n}}$ for an $\hat{n} \in \mathbb{N}$,

because the union of all subspaces E_n is dense. The sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converges monotonically towards zero. So there is an $N_0 \in \mathbb{N}$ with $\varepsilon_{n-1}\|x\| + \varepsilon_{n-1} + \varepsilon_n \le \varepsilon$ for all $n > N_0$. Then we get for all $n > \max\{\hat{n}, N_0\}$ using (2.1) and $m_{n-1} \ge n-1$

$$\|x\| + 1 \ge \|x + f_n\| \ge \left\| x + \sum_{k=m_{n-1}+1}^{m_n} \lambda_k e_k \right\| - \left\| \sum_{k=m_{n-1}+1}^{m_n} \lambda_k e_k + f_n \right\|$$

$$\ge (1 - \varepsilon_{n-1})\|x\| + \sum_{k=m_{n-1}+1}^{m_n} (1 - \varepsilon_{k-1})|\lambda_k| - \varepsilon_n$$

$$\ge (1 - \varepsilon_{n-1})\|x\| + (1 - \varepsilon_{n-1}) \sum_{k=m_{n-1}+1}^{m_n} |\lambda_k| - \varepsilon_n$$

$$\ge (1 - \varepsilon_{n-1})\|x\| + (1 - \varepsilon_{n-1}) - \varepsilon_n$$

$$\ge \|x\| + 1 - \varepsilon.$$

So $\lim_{n\to\infty} \|x + f_n\| = \|x\| + 1$ and $(f_n)_{n\in\mathbb{N}}$ is a canonical ℓ_1-type sequence in Z. $\qquad\square$

References

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